

DOUBLE CENTRALIZERS IN ARTIN-TITS GROUPS

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ABSTRACT. We prove an analogue of the Centralizer Theorem in the context of Artin-Tits groups.

INTRODUCTION

To obtain information on a group G , a standard approach consists in considering subgroups and studying how they behave in the group. In particular, one often consider the centralizer $Z_G(H)$ of a subgroup H in G , which is defined by

$$Z_G(H) = \{g \in G \mid gh = hg \text{ for all } h \in H\}.$$

This general approach naturally extends to other contexts. This is the case in the study of noncommutative algebras where subgroups are replaced by subalgebras. Clearly, for an algebra R and a subalgebra H , the centralizer $Z_G(H)$ is also a subalgebra. In this framework, the subalgebra $Z_G(Z_G(H))$, called the *double centralizer* of H , has been considered [10, 25]. For instance, a classical result [10] is the so-called *Centralizer Theorem*, which claims that for a finite dimensional central simple algebra R over a field k and for a simple subalgebra H , one has $Z_G(Z_G(H)) = H$. Various generalizations has been obtained leading to applications [26, 6].

Regarding the result obtained in the algebra framework, and coming back to the group theory framework, one is naturally lead to consider the double-centralizer subgroup $Z_G(Z_G(H))$ of a subgroup H in a group G and to address the question of a similar Centralizer Theorem. Let us denote by $DZ_G(H)$ the double centralizer of H . Obviously, when the group G has a center $Z(G)$ that is not contained in the subgroup H , the equality $DZ_G(H) = H$ can not hold. However, one may wonder whether the subgroup $DZ_G(H)$ is generated by $Z(G)$ and H . More precisely, if $Z(G) \cap H$ is trivial, one may wonder whether $DZ_G(H) = Z(G) \times H$. When the center of G is trivial, we recover the property of the Centralizer Theorem namely, $DZ_G(H) = H$.

As far as we know, the first Centralizer Theorem in the group theory framework has been obtained in [12] by considering the braid group on n strands and its standard parabolic subgroups. Our objective here is to address the more general case of an Artin-Tits group G and a standard parabolic subgroup H . *Artin-Tits* groups are those groups which possess a presentation associated with a Coxeter matrix. For a finite set S , a Coxeter matrix on S is a symmetric matrix $(m_{s,t})_{s,t \in S}$ whose entries are either a positive integer or equal to ∞ , with $m_{s,t} = 1$ if and only if $s = t$. An Artin-Tits group associated with such a matrix is defined by the presentation

$$(1) \quad \left\langle S \mid \underbrace{sts \dots}_{m_{s,t} \text{ terms}} = \underbrace{tst \dots}_{m_{s,t} \text{ terms}} ; \forall s, t \in S, s \neq t ; m_{s,t} \neq \infty \right\rangle.$$

For instance, If we consider $S = \{s_1, \dots, s_n\}$ with $m_{s_i, s_j} = 3$ for $|i - j| = 1$ and $m_{s_i, s_j} = 2$ otherwise, we obtain the classical presentation of the braid group B_{n+1} on $n + 1$ strings considered in [12]. A *standard parabolic subgroup* is a subgroup generated by a subset X of S . It turns out that such a subgroup is also an Artin-Tits groups in a natural way (see Proposition 1.1 below). Artin-Tits groups are badly understood and most articles on the subject focus on particular subfamilies of Artin-Tits groups, such as Artin-Tits groups of spherical type, of FC type, of large type, or of 2-dimensional type. Here again, we apply this strategy. We first consider the family of spherical type Artin-Tits groups, whom seminal example are braid groups. We refer to the next sections for definitions. We prove:

Theorem 0.1. *Assume A_S is a spherical type irreducible Artin-Tits group with S for standard generating set. Let X be strictly included in S and A_X be the standard parabolic subgroup of A generated by X . Denote by Δ the Garside element of A_S .*

- (i) *If Δ lies in $DZ_{A_S}(A_X)$ but not in $Z(A_S)$, then*

$$DZ_{A_S}(A_X) = A_X \times QZ(A_S)$$

- (ii) *If not,*

$$DZ_{A_S}(A_X) = A_X \times Z(A_S).$$

In the above result we do not consider the case $X = S$. Indeed, for any group G one has $DZ_G(G) = G$. In the present article, we also consider Artin-Tits groups that are not of spherical type. We conjecture that

Conjecture 0.2. *Assume A_S is an irreducible Artin-Tits group. Let A_X be a standard parabolic subgroup of A_S generated by a subset X of S . Assume A_X is irreducible. Let A_T be the smallest standard parabolic subgroup of A_S that contains $Z_{A_S}(A_X)$.*

- (i) *Assume A_X is not of spherical type. Then $DZ_{A_S}(A_X) = Z_{A_S}(A_T)$.*
(ii) *Assume A_X is of spherical type.*
(a) *if A_T is of spherical type, then,*

$$DZ_{A_S}(A_X) = DZ_{A_T}(A_X).$$

- (b) *If A_T is not of spherical, then*

$$DZ_{A_S}(A_X) = A_X.$$

The centralizer of a standard parabolic subgroup is well-understood in general. In particular, when Conjectures 1, 2, 3 of [17] hold, for any given X , one can read on the Coxeter graph Γ_S whether or not the above group A_T is of spherical type. This is the case for the Artin-Tits groups considered in Theorem 0.3. The conjecture is supported by the following result:

Theorem 0.3. (i) *Conjecture 0.2 holds for irreducible Artin-Tits groups of FC type.*

- (ii) *Conjecture 0.2 holds for Artin-Tits groups of 2-dimensional type.*

- (iii) *Conjecture 0.2 holds for Artin-Tits groups of large type.*

The reader may note that in Theorem 0.1 there is no restriction on A_X , whereas in Conjecture 0.2 we assume that A_X is irreducible. Indeed, We can extend the above conjecture to the case where X not irreducible (see Conjecture 3.2) and prove that this general conjecture holds for the same Artin-Tits groups than those considered in Theorem 0.3. However, the statement is more technical. This is

why we postpone it and restrict to the irreducible case in the introduction. The remainder of this article is organized as follows. In Section 2, we introduce the necessary definitions and preliminaries. Section 3 is devoted to Artin-Tits groups of spherical type. Finally, in Section 4, we turn to the not spherical type cases.

1. PRELIMINARIES

In this section we introduce the useful definitions and results on Artin-Tits groups that we shall need when proving our theorems. For all this section, we consider an Artin-Tits group A_S generated by a set S and defined by Presentation (1) given in the introduction.

1.1. Parabolic subgroups. As explained, the subgroups that we consider in the article are the so-called *standard parabolic subgroups*, that is those subgroups that are generated by a subset of S . One of the main reasons that explains why these subgroups are considered is that they are themselves Artin-Tits groups:

Proposition 1.1. [27] *Let X be a subset of S . Consider the Artin-Tits group A_X associated with the Coxeter matrix $(m_{st})_{s,t \in X}$. Then*

- (i) *the canonical morphism from A_X to A_S that sends x to x is into. In particular, A_X is isomorphic to, and will be identified with, the subgroup of A_S generated by X .*
- (ii) *if Y is another subset of S , then we have $A_X \cap A_Y = A_{X \cap Y}$.*

We have already defined the notion of a centralizer $Z_{A_S}(A_X)$ of a subgroup A_X . We recall that we denote the center $Z_{A_S}(A_S)$ of A_S by $Z(A_S)$. More generally, for a subset X of S , by $Z(A_X)$ we denote the center of the parabolic subgroup A_X . Along the way, we will also need the notions of a normalizer of a subgroup and of a quasi-centralizer of a parabolic subgroups. We recall here their definitions.

Definition 1.2. Let X be a subset of S and A_X be the associated standard parabolic subgroup.

- (i) The *normalizer* of A_X in A_S , denoted by $N_{A_S}(A_X)$, is the subgroup of A_S defined by

$$N_{A_S}(A_X) = \{g \in A_S \mid g^{-1}A_Xg = A_X\}$$

- (ii) The *quasi-centralizer* of A_X in A_S , denoted by $QZ_{A_S}(A_X)$, is the subgroup of A_S defined by

$$QZ_{A_S}(A_X) = \{g \in A_S \mid g^{-1}Xg = X\}$$

In the sequel, we will write $QZ(A_S)$ for $QZ_{A_S}(A_S)$. There is an obvious sequence of inclusion between these subgroups:

$$Z_{A_S}(A_X) \subseteq QZ_{A_S}(A_X) \subseteq N_{A_S}(A_X).$$

But we can say more:

Theorem 1.3. [15, 16, 17] *Let A_S be an Artin-Tits group, and X be a subset of S . If A_S is of spherical type or of FC type or of 2-dimensional type, then*

$$N_{A_S}(A_X) = QZ_{A_S}(A_X) \cdot A_X.$$

This result is one of the key arguments in our proof of Theorems 0.1 and 0.3. Actually, it is conjectured in [13] that this property holds for any Artin-Tits groups.

1.2. Families of Artin-Tits groups. Our objective now is to introduce the various families of Artin-Tits groups that we considered in the introduction.

1.2.1. Irreducible Artin-Tits groups. First, we say that an Artin-Tits group is *irreducible* when it is not the direct product of two of its standard parabolic subgroups. Otherwise we say that it is *reducible*. Associated with the Coxeter matrix $(m_{s,t})_{s,t \in S}$ is the Coxeter graph, which is the simple labelled graph with S as vertex set defined as it follows. There is an edge between two distinct vertices s and t when $m_{s,t}$ is not two. The edge has label $m_{s,t}$ when $m_{s,t}$ is not 3. Therefore, the group A_S is irreducible if and only if the Coxeter graph Γ_S is connected. For instance the braid group on $n + 1$ strings is irreducible whereas the free abelian group on two generators is not.

1.2.2. Spherical type Artin-Tits groups. Among Artin-Tits groups, those of spherical type are the most studied and the most understood. From Presentation (1), we obtain the presentation of the associated Coxeter group by adding the relations $s^2 = 1$ for s in S . The Artin-Tits group is said to be of spherical type when this associated Coxeter group is finite. For instance, braid groups are of spherical type as their associated Coxeter groups are the symmetric groups. Actually there is only a finite list of connected Coxeter graphs whom associated (irreducible) Artin-Tits groups are of spherical type (see [7],[2]).

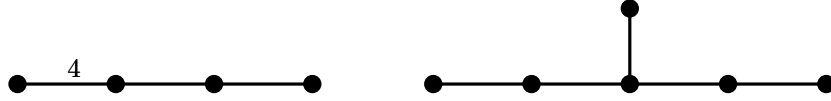


FIGURE 1. Artin-Tits groups of spherical types $B(4)$ and $E(6)$

1.2.3. FC type Artin-Tits groups. These Artin-Tits groups are built on those of spherical type. An Artin-Tits group is of FC type when all its standard parabolic subgroups whom Coxeter graphs have no edge labelled with ∞ are of spherical type. In particular, all spherical type Artin-Tits groups are of FC type. Alternatively, the family of FC type Artin-Tits groups can be defined as the smallest family of groups that contains spherical type Artin-Tits groups and that is closed under amalgamation above a standard parabolic subgroup. For instance, the Artin-Tits group associated with the following Coxeter graph is of FC type.

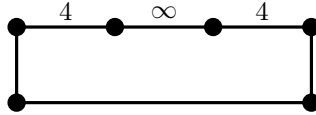


FIGURE 2. A FC type Artin-Tits group

Indeed, the Artin-Tits group in Figure 2 is the amalgamation of two spherical type Artin-Tits groups of type $B(5)$ (see [1]) above a common standard parabolic subgroup, which is of type $A(4)$, that is a braid group B_5 .

1.2.4. *2-dimensional type Artin-Tits groups.* An Artin-Tits group is of 2-dimensional type when no standard parabolic subgroup generated by three, or more, generators is of spherical type. These groups has been considered, for instance, in [4, 5, 17].

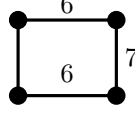


FIGURE 3. A 2-dimensional Artin-Tits group

1.2.5. *Large type Artin-Tits groups.* Contained in the family of 2-dimensional Artin-Tits groups is the family of Artin-Tits groups of large type. An Artin-Tits group is of large type when no $m_{s,t}$ is equal to 2. Some 2-dimensional Artin-Tits groups are not of large type (see Figure 3).



FIGURE 4. Artin-Tits groups of large types $\tilde{A}(2)$ and $I(5)$.

1.3. **Artin-Tits monoids.** As explained above, one of the main ingredients in our proof is Theorem 1.3. Another one is the positive monoid of an Artin-Tits monoid that allows to apply Garside theory. Here, we introduce only the results that we will need and refer to [8] for more details on this theory. We recall that we fix an Artin-Tits group A_S generated by a set S and defined by Presentation (1).

Definition 1.4. The Artin-Tits monoid A_S^+ associated with A_S is the submonoid of A_S generated by S . An element of A_S that belongs to A_S^+ is called a positive element. Its inverse is called a negative element.

We gather in the following proposition several properties of Artin-Tits monoids that we will need in the sequel.

Proposition 1.5. (i) [24] *Considered as a presentation of monoid, Presentation (1) is a presentation of the monoid A_S^+ .*

(ii) *When A_S is of spherical type, then*

- (a) [2, 3, 8] *the monoid A_S^+ is a Garside monoid. In particular, every element g in A_S can be decomposed in a unique way as $g = a^{-1}b$, with a, b positive, so that a and b have no nontrivial common left-divisors in A_S^+ . Furthermore, if $c \in A_S^+$ is such that $cg \in A_S^+$, then a right-divides c in A_S^+ .*
- (b) [2, 9] *There is a positive element Δ that belongs to $QZ(A_S)$ so that every element g in A_S , can be decomposed as $g = a\Delta^{-n}$ with a positive and $n \geq 0$. Moreover, Δ^2 belongs to $Z(A_S)$.*

- (c) [2, 9] When, moreover, A_S is irreducible then $QZ(A_S)$ is an infinite cyclic group generated by Δ . The group $Z(A_S)$ is infinite cyclic generated by Δ or by Δ^2 .

The decomposition $g = a^{-1}b$ in Point (ii)(a) is called the *Charney's (left) orthogonal splitting* of g . The *Charney's right orthogonal splitting* $g = ab^{-1}$ is defined in a similar way.

In the sequel, we denote by $\tau : S \rightarrow S$ the permutation of S defined by $\Delta s = \tau(s)\Delta$ for all s in S . As explained above, τ is either the identity or an involution. In particular, we have also $s\Delta = \Delta\tau(s)$ for all s in S . Moreover, for a, b in A_S^+ , we write $a \preceq b$ if a left-divides b in A_S^+ , that is if there exists c in A_S^+ so that $b = ac$. Similarly, we write $b \succeq a$ if a right-divides b in A_S^+ .

2. SPHERICAL TYPE ARTIN-TITS GROUPS

In this Section we focus on spherical type Artin-Tits groups and prove Theorem 0.1.

2.1. Artin-Tits groups of type $E(6)$ and $D(2k+1)$. In Theorems 0.1 the description of $DZ_{A_S}(A_T)$ depends on a technical condition. Here we investigate this condition and characterize irreducible Coxeter graphs for which this condition is satisfied.

Proposition 2.1. *Assume A_S is an irreducible spherical type Artin-Tits group. Let X be a proper subset of S . Then, Δ does not belong to $Z(A_S)$ but lies in $DZ_{A_S}(A_X)$ if and if :*

- (a) *either Γ_S is of type $D(2k+1)$ and $X \supseteq \{s_2, s_{2'}, s_3\}$ (see Figure 5).*
- (b) *or Γ_S is of type E_6 and $X = \{s_2, \dots, s_6\}$ (see Figure 6).*

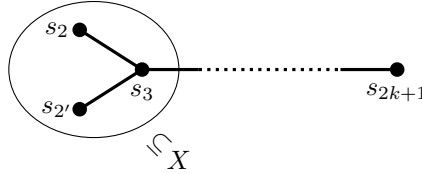


FIGURE 5. Γ_S of type $D(2k+1)$ and $X \supseteq \{s_2, s_{2'}, s_3\}$

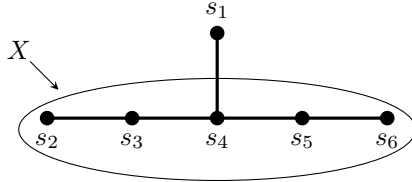


FIGURE 6. Γ_S of type E_6 and $X = \{s_2, \dots, s_6\}$

When proving Proposition 2.1, we will need the following lemma.

Lemma 2.2. *Assume A_S is an irreducible spherical type Artin-Tits group. Let X be a proper subset of S . Assume the permutation τ is not the identity on S and Δ lies in $DZ_{A_S}(A_X)$ then:*

- (i) τ is the identity on $S \setminus X$, that is Δ lies in $Z_{A_S}(A_{S \setminus X})$.
- (ii) τ is not the identity on X , that is Δ does not lie in $Z_{A_S}(A_X)$.
- (iii) Δ stabilizes the indecomposable components of X .

Proof. (i) Let $s \in S \setminus X$. Set $Y = X \cup \{s\}$. The elements Δ_X^2, Δ_Y^2 lie in $Z(A_X)$ and $Z(A_Y)$, respectively. So, they both belong to $Z_{A_S}(A_X)$ and, therefore, commute with Δ . Since $\Delta\Delta_X = \Delta_{\tau(X)}\Delta$ and $\Delta\Delta_Y = \Delta_{\tau(Y)}\Delta$, we deduce that $\tau(X) = X$ and $\tau(Y) = Y$. Using that $Y = X \cup \{s\}$, we concluded that $\Delta s = s\Delta$. Thus, Δ lies in $Z_{A_S}(A_{S \setminus X})$ and (i) holds. Since τ is not the identity on S , (i) implies (ii). Finally, Let X_1 be an indecomposable component of X . We have $\Delta\Delta_{X_1} = \Delta_{\tau(X_1)}\Delta$. Then, $\Delta_{X_1}^2$ lies in $Z(A_X)$ and, therefore, in $Z_{A_S}(A_X)$. Hence $\Delta\Delta_{X_1}^2 = \Delta_{X_1}^2\Delta$ and $X_1 = \tau(X_1)$, that is $\Delta X_1 = X_1\Delta$. \square

Proof of Proposition 2.1. Assume the element Δ does not belong to the center $Z(A_S)$ but lies in $DZ_{A_S}(A_X)$. So, assertions (i)(ii) and (iii) in Lemma 2.2 hold. In particular, the permutation τ is not the identity map on S . Using the classification of irreducible Artin-Tits groups [1] and well-known results on Δ [2, 9], we deduce that the type of Γ_S is one of the following types:

- $A(k)$ with $k \geq 2$,
- $D(2k+1)$ with $k \geq 1$,
- E_6 , or
- $I_2(2p+1)$ with $p \geq 1$.

By Lemma 2.2(i), the permutation τ fixes each element of $S \setminus X$. This imposes that Γ_S cannot be of type $I_2(2p+1)$, as X is proper in S and Δ permutes the two elements of S . If Γ_S is of type $A(k)$ with $k \geq 2$, (so A_S is the braid group B_{k+1}) then the unique element of S fixed by τ is $s_{\frac{k+1}{2}}$. This imposes $S \setminus X = \{s_{\frac{k+1}{2}}\}$. and Δ does not fix the two indecomposable components $\{s_1, \dots, s_{\frac{k-1}{2}}\}$ and $\{s_{\frac{k+3}{2}}, \dots, s_k\}$ of X , a contradiction with Lemma 2.2(iii). So Γ_S is not of type $A(k)$. If Γ_S is of type $D(2k+1)$, then τ switches s_2 and $s_{2'}$. Therefore, s_2 and $s_{2'}$ have to lie in X . Moreover, s_2 does not commute with Δ , so it cannot belong to $Z_{A_S}(A_X)$. This imposes that s_3 has to belong to X . Hence, $\{s_2, s_{2'}, s_3\}$ is included in X and we have case (a) of the proposition. Similarly, if Γ_S is of type E_6 . The elements s_2, s_3, s_5, s_6 are not fixed by τ , so they have to belong to X . Applying Lemma 2.2(iii), we deduce that s_4 has to lie in X too. Since X is not S , it is equal to $\{s_2, \dots, s_6\}$ and we have case (b) of the proposition. Conversely, in case (a) and (b), one can verify that Δ does not belong to $Z(A_S)$ but lies on $DZ_{A_S}(A_X)$ \square

2.2. Ribbons. The notion of ribbon introduced in [11] for the case of braid groups, and then generalized in [23, 14], will be crucial to us in order to calculate the double-centralizer of a parabolic subgroup. Here we recall its definition and gather some properties that we shall need. In particular, we only consider spherical type Artin-Tits groups. We refer to above references and to [8] for more details. Given an Artin-Tits presentation (1), let us first introduce two notations: for a subset X of S , we set

$$X^\perp = \{s \in S \setminus X \mid \forall t \in X, m_{ts} = 2\}$$

and

$$\partial X = \{s \in S \setminus X \mid \exists t \in X, m_{ts} > 2\}.$$

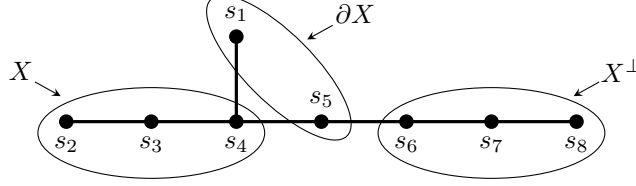


FIGURE 7. Example : $\partial(X)$ and X^\perp

Definition 2.3. (i) Let t belong to S and X be included in S . Denote by $X(t)$ the indecomposable component of $X \cup \{t\}$ containing t . If t lies in X , we set $d_{X,t} = \Delta_{X(t)}$; otherwise, we set

$$d_{X,t} = \Delta_X \Delta_{X-\{t\}}^{-1},$$

that is $d_{X,t} = \Delta_{X(t)} \Delta_{X(t)-\{t\}}^{-1}$. In both cases, there exists $Y \subseteq X \cup \{t\}$ and $t' \in X(t)$ so that $Y d_{X,t} = d_{X,t} X$ with $Y \cup \{t'\} = X \cup \{t\}$ and $Y(t') = X(t)$. The element $d_{X,t}$ is called a *positive elementary Y-ribbon-X*.

(ii) For $X, Y \subseteq S$, we say that $g \in A_S^+$ is a positive Y -ribbon- X if $Yg = gX$.

For instance, considering the example in Figure 7, $d_{X,s_5} = s_2 s_3 s_4 s_5$, $d_{X,t} = t$ for t in X^\perp and Δ is a positive X -ribbon- X .

The connection between positive ribbons and elementary ones appears in the following result

Proposition 2.4. Assume A_S is a spherical type Artin-Tits group and g lie in A_S^+ . g is a positive Y -ribbon- X if and only if $g = g_n \cdots g_1$ where each g_i is a positive elementary X_i -ribbon- X_{i-1} , with $X_0 = X$ and $X_n = Y$.

Proposition 2.5. Assume A_S is a spherical type Artin-Tits group. Let X be included in S and u be included in A_S^+ . Let $\varepsilon \in \{1, 2\}$ be such that Δ_X^ε lies in $Z(A_X)$.

- (i) Assume u is a positive Y -ribbon- X for some $Y \subseteq S$.
 - (a) $\Delta_Y u = u \Delta_X$.
 - (b) Assume t belongs to S . Then, $u \succeq t \Leftrightarrow u \succeq d_{X,t}$.
- (ii) If $u \Delta_X^\varepsilon \succeq u$, then there exists $Y \subseteq S$ such that $u \Delta_X^\varepsilon u^{-1} = \Delta_Y^\varepsilon$, $u A_X u^{-1} = A_Y$ and $\Gamma_X \sim \Gamma_Y$. Moreover, if u is reduced- X , then u is a positive Y -ribbon- X , that is $Y u = u X$.

The above results are not all explicitly stated in [23, 14] but are well-known from specialists. The second part of (ii) is stated in [14, Lemma 2.2] and the first part follows (see also [23, Lemma 5.6]). Point (i) is proved in the proof of [14, Lemma 2.2] (see [23, Lemma 5.6] for details). For point (i)(b), see also [18, Example 3.14]. The *support* of a word on D is the set of letters that are involved in this word. It follows from the presentation of A_S^+ that two representing words of the same element in A_S^+ have the same support. So the *support* of an element of A_S^+ is well-defined. In the sequel, by $Supp(g)$ we denote the support of an element g in A_S^+ .

Lemma 2.6. *Assume A_S is a spherical type Artin-Tits group. Let $X \subsetneq S$ be such that Γ_X is connected, and $t \in \partial X$. Then*

$$\text{Supp}(d_{X,t}) = X \cup \{t\}.$$

Proof. By assumption t is not in S , so $d_{X,t} = \Delta_X \Delta_{X-\{t\}}^{-1}$ and $\text{Supp}(d_{X,t})$ is included in $X \cup \{t\}$. Let us show the converse inclusion. Now, by Proposition 2.5 (i), we have $d_{X,t} = v_0 s_0$ for some v_0 in A_S^+ and $s_0 = t$ belongs to the support of $d_{X,t}$. Let s be in X . Since X is connected and t belongs to ∂X , there exists a finite sequence s_1, \dots, s_n of X such that $s_n = s$ and for all $i \geq 0$, we have $m_{s_i, s_{i+1}} \neq 2$. We assume the sequence is chosen so that n is minimal. Assume $d_{X,t} = v_i s_i \cdots s_0$ for some $0 \leq i < n$ with v_i in A_S^+ . Since $Y d_{X,t} = d_{X,t} X$ for some $Y \subseteq X \cup \{t\}$, we can write $v_i s_i \cdots s_0 s_{i+1} = s'_{i+1} v_i s_i \cdots s_0$ for some s'_{i+1} in $X \cup \{t\}$. By minimality of n , $m_{s_j, s_{j+1}} = 2$ for any $j < i$. So $v_i s_i s_{i+1} s_{i-1} \cdots s_0 = s'_{i+1} v_i s_i \cdots s_0$ and $v_i s_i s_{i+1} = s'_{i+1} v_i s_i$. This imposes that $v_i s_i s_{i+1} = s'_{i+1} v_i s_i = \underbrace{v' \cdots s_{i+1} s_i s_{i+1}}_{m \text{ terms}}$

with $m = m_{s_i, s_{i+1}}$ and v' in A_S^+ (see [9, 2]). This imposes in turn that we can write $v_i = v_{i+1} s_{i+1}$ and $d_{X,t} = v_i s_i \cdots s_0$. Then, we obtain step-by-step that $d_{X,t}$ can be decomposed as $v_n s_n \cdots s_0$. Hence s belongs to the support of $d_{X,t}$ for any s in X . So the converse inclusion holds. \square

Lemma 2.7. *Let $u \in A_S^+$ and $s \in S$. Denote by $u_2^{-1} v_1$ the left orthogonal splitting of the element $u^{-1} s u$. There exists u_1 in A_S^+ and s_1 in S so that $u = u_1 u_2$, $v_1 = s_1 u_2$. Moreover, u_1 is a positive s -ribbon- s_1 .*

Proof. By [19, Theorem 1] there exists u_1 in A_S^+ so that $u = u_1 u_2$ and $v_1 = s_1 u_2$ for some s_1 in S . Moreover, applying [19, Lemma 2.3], a straightforward induction on the length of u proves that u_1 is a positive s -ribbon- s_1 . \square

In the sequel, we say that an element of A_S^+ is a positive ribbon- X when it is a positive Y -ribbon- X , for some Y . Similarly we say that an element is a positive Y -ribbon when it is a positive Y -ribbon- X .

2.3. The proof of Theorem 0.1. In this section we prove Theorem 0.1. The proof need two preliminary results, namely Lemma 2.8 and Proposition 2.9, which is the main argument. The latter is proved here; the proof of the former is postponed to the next section.

Lemma 2.8. *Under the assumptions of Proposition 2.9, we have $b \succeq s$ for all $s \in S \setminus X$.*

Proposition 2.9. *Let A_S be an irreducible Artin-Tits group of spherical type. Let $X \subsetneq S$. Let b be in $A_S^+ \setminus \{1\}$ a positive ribbon- $(X \cup X^\perp)$ that is reduced- X . Suppose further that for all $Y \subseteq S$ containing X , and $\varepsilon(Y) \in \{1, 2\}$ minimal such that $\Delta_Y^{\varepsilon(Y)} \in Z_{A_S}(A_X)$, we have $b \Delta_Y^{\varepsilon(Y)} \succeq b$. Then there exists $n \in \mathbb{N}^*$ so that*

$$b = \Delta^n \Delta_X^{-n}$$

Note that Δ_X right-divides Δ in A_S^+ and $\Delta \Delta_X = \Delta_{\tau(X)} \Delta$ by Proposition 1.5. So for any positive integer n the element $\Delta^n \Delta_X^{-n}$ belongs to A_S^+ .

Proof of Proposition 2.9. We have $\Delta_X \succeq s$ for all $s \in X$ and, by Lemma 2.8, we have $b \succeq s$ for all $s \in S \setminus X$. Since, by assumption, $b \Delta_X \succeq b$, we get that $b \Delta_X \succeq s$ for all $s \in S$. Thus $b \Delta_X \succeq \Delta$ and, therefore, $b \succeq \Delta \Delta_X^{-1}$ in A_S^+ . Let $k \in \mathbb{N}^*$

be maximal such that $b \succeq \Delta^k \Delta_X^{-k}$. Write $b = d\Delta^k \Delta_X^{-k}$ with $d \in A_S^+$. We show that $d = 1$. This will prove the proposition. Since $\Delta X = \tau(X)\Delta$, the element Δ^k is a positive $\tau^k(X)$ -ribbon- X and $\Delta^k X = \tau^k(X)\Delta^k$. Therefore by Proposition 2.5, we have $\Delta^k \Delta_X = \Delta_{\tau^k(X)} \Delta^k$. For the remaining of the proof, for $Z \subseteq S$, we set $Z_k = \tau^k(Z)$. Moreover, Δ_X is a positive X -ribbon- X . Then $\Delta^k \Delta_X^{-k}$ is a positive X_k -ribbon- X . For the remaining of the proof, when s lies in X_k , we denote by s_X the element of X so that $s\Delta^k \Delta_X^{-k} = \Delta^k \Delta_X^{-k} s_X$.

Assume there exists s in X_k so that $d = us$ with u in A_S^+ . Then we have $b = us\Delta^k \Delta_X^{-k}$. We get $b = u\Delta^k \Delta_X^{-k} s_X$. But this is not possible, since b is reduced- X . Hence, d is reduced- X_k . We now prove that d is a positive ribbon- $(X_k \cup X_k^\perp)$. Let s lie in X_k . We have $s\Delta^k \Delta_X^{-k} = \Delta^k \Delta_X^{-k} s_X$. By assumption, b is a positive ribbon- X , therefore there exists s' in S so that $bs_X = s'b$. Hence we get $ds\Delta^k \Delta_X^{-k} = d\Delta^k \Delta_X^{-k} s_X = s'd\Delta^k \Delta_X^{-k}$, and therefore $ds = s'd$. As this so for every element of X_k , we deduce that d is a positive ribbon- X_k . Let s lie X_k^\perp . For every t in X , we have $\tau^k(t)$ lies in X_k and, therefore, $m_{\tau^k(t),s} = 2$. But the involution τ induces an automorphism of the Coxeter graph associated with the presentation of A_S . It follows that for every t in X , we have $m_{t,\tau^k(s)} = m_{\tau^k(t),s} = 2$. Hence, $\tau^k(s)$ belongs to X^\perp . But b is a positive ribbon- X^\perp , then $b\tau^k(s) = s'b$ for some $s' \in S$. Hence we get $ds\Delta^k \Delta_X^{-k} = d\Delta^k \Delta_X^{-k} \tau^k(s) = s'd\Delta^k \Delta_X^{-k}$, and therefore $ds = s'd$. As this so for every element of X_k^\perp , we deduce that d is a positive ribbon- X_k^\perp . Gathering the two results we get that d is a positive ribbon- $(X_k \cup X_k^\perp)$.

Let $Y \subseteq S$ containing X_k and consider $\eta(Y)$ be positive and minimal such that $\Delta_Y^{\eta(Y)}$ belongs to $Z_{A_S}(A_{X_k})$. The involution τ^k exchanges X and X_k and exchanges Y and Y_k . It follows, Firstly, that the inclusion, $X_k \subseteq Y$ implies the inclusion $X \subseteq Y_k$ and, Secondly, that τ^k send A_{X_k} and Δ_Y to A_X and Δ_{Y_k} , respectively, with $\eta(Y) = \varepsilon(Y_k)$. Thus, $\Delta_{Y_k}^{\eta(Y)}$ belongs to $Z_{A_S}(A_X)$ with $\eta(Y) = \varepsilon(Y_k)$. Then, by assumption, we have $b\Delta_{Y_k}^{\eta(Y)} = ub$, for some u in A_S^+ . Since $b\Delta_{Y_k}^{\eta(Y)} = d\Delta^k \Delta_X^{-k} \Delta_{Y_k}^{\eta(Y)} = d\Delta^k \Delta_{Y_k}^{\eta(Y)} \Delta_X^{-k} = d\Delta_Y^{\eta(Y)} \Delta^k \Delta_X^{-k}$ and $ub = ud\Delta^k \Delta_X^{-k}$ we obtain that $d\Delta_Y^{\eta(Y)} = ud$. As a consequence, replacing b and X by d and X_k , respectively, we can repeat the beginning of the argument and deduce that $d = d_1\Delta\Delta_{X_k}^{-1}$ for some d_1 in A_S^+ . But this lead to a contradiction to the maximality of k , since we get $b = d\Delta^k \Delta_X^{-k} = d_1\Delta\Delta_{X_k}^{-1}\Delta^k \Delta_X^{-k} = \Delta^{k+1}\Delta_X^{-(k+1)}$. Hence $d = 1$ and $b = \Delta^k \Delta_X^{-k}$. \square

We turn now to the proof of Theorem 0.1.

Proof of Theorem 0.1. Let u lie in $DZ_{A_S}(A_X)$. We have $Z_{A_X}(A_X) \subseteq Z_{A_S}(A_X)$, then $DZ_{A_S}(A_X) \subseteq Z_{A_S}(Z_{A_X}(A_X))$ and u belongs to $Z_{A_S}(Z_{A_X}(A_X))$. Thanks to Theorem 1.3, we can write $u = y \cdot z$, with $yX = Xy$ and $z \in A_X$. Write (see Proposition 1.5) $y = \Delta^{-2m}h$ with h in A_S^+ , and decompose h as $h = abc$, with $a, c \in A_X^+$ and b being X -reduced- X . Since $yX = Xy$ and Δ^2 is in $Z(A_S)$, we have $hX = Xh$ and so $h\Delta_X = \Delta_X h$. Using that $h = abc$ with a, c in A_X^+ and that Δ_X^2 lie in $Z(A_X)$, we deduce that $b\Delta_X^2 = \Delta_X^2 b$. The element b is reduced- X , then by Proposition 2.5, we have $bX = Xb$. It follows there exists $z' \in A_X$ such that $bcz = z'b$. Set $x = az'$. Then, x belongs to A_X and $u = \Delta^{-2m} \cdot x \cdot b$. Suppose $b \neq 1$. By definition $X^\perp \subseteq Z_{A_S}(A_X)$. Therefore, for all $s \in X^\perp$ we have $us = su$ and $s\Delta^{-2m} \cdot x \cdot b = \Delta^{-2m} \cdot x \cdot sb$. By cancellation, we obtain $bs = sb$

for all $s \in X^\perp$. So, b is a positive ribbon- $(X \cup X^\perp)$. Now, let Y be included in S and containing X . Set $\varepsilon(Y) \in \{1, 2\}$ be minimal such that $\Delta_Y^{\varepsilon(Y)}$ lies in $Z_{A_S}(A_X)$. Then, $u\Delta_Y^{\varepsilon(Y)} = \Delta_Y^{\varepsilon(Y)}u$ and, as before, we get $b\Delta_Y^{\varepsilon(Y)} = \Delta_Y^{\varepsilon(Y)}b$. As a consequence we have $b\Delta_Y^{\varepsilon(Y)} \succeq b$. By Proposition 2.9, we deduce there exists $n \in \mathbb{N}^*$ so that $b = \Delta^n \Delta_X^{-n}$. Thus, we get $u = \Delta^{-2m}x\Delta^n \Delta_X^{-n}$.

Assume, First, that $\Delta \in DZ_{A_S}(A_X)$ and $\Delta \notin Z(A_S)$. Then, by Lemma 2.2, we have $\Delta X = X\Delta$ and $\tau^n(x)$ belongs to A_X . Therefore, $u = \Delta^{-2m+n} \cdot \tau^n(x) \Delta_X^{-n}$ and u belongs to $QZ(A_S) \cdot A_X$. So $DZ_{A_S}(A_X)$ is included in $QZ(A_S) \cdot A_X$. Conversely, The assumption that Δ lies in $DZ_{A_S}(A_X)$ imposes the inclusion $QZ(A_S) \cdot A_X \subseteq DZ_{A_S}(A_X)$. Therefore, the latter inclusion is actually an equality. Moreover we have $QZ(A_S) \cdot A_X = A_X \cdot QZ(A_S)$, since Δ belongs to $QZ_{A_S}(A_X)$ by the above argument.

Assume, Secondly that $\Delta \notin DZ_{A_S}(A_X)$ or $\Delta \in Z(A_S)$. First, the inclusion $Z(A_S) \cdot A_X \subseteq DZ_{A_S}(A_X)$ holds in any case. If $\Delta \in Z(A_S)$ or n is even, then u , that is $\Delta^{-2m+n}x\Delta_X^{-n}$, lies in $Z(A_S) \cdot A_X$ and so the other inclusion $DZ_{A_S}(A_X) \subseteq Z(A_S) \cdot A_X$ holds too. Assume finally $\Delta \notin DZ_{A_S}(A_X)$. Since u lies in $DZ_{A_S}(A_X)$, for every w in $Z_{A_S}(A_X)$ we have $wu = uw$ and therefore $\Delta^{-2m}xw\Delta^n \Delta_X^{-n} = \Delta^{-2m}x\Delta^n w\Delta_X^{-n}$. This, in turn, imposes $\Delta^n w = w\Delta^n$ for every w in $Z_{A_S}(A_X)$. In other words Δ^n lies in $DZ_{A_S}(A_X)$ too. Since Δ^2 lies in $DZ_{A_S}(A_X)$ but not Δ , we deduce that n has to be even, and conclude by the above argument that $Z(A_S) \cdot A_X = DZ_{A_S}(A_X)$.

Finally, we note that $A_X \cap QZ(A_S) = A_X \cap Z(A_S) = \{1\}$. Indeed, $X \neq S$ and $\text{Supp}(\Delta) = S$. Therefore, Δ^m does not belong to A_X except if $m = 0$. Hence, we have $A_X \cdot QZ(A_S) = A_X \times QZ(A_S)$ and $A_X \cdot Z(A_S) = A_X \times Z(A_S)$. \square

2.4. The proof of Lemma 2.8. Here we focus on the proof of Lemma 2.8, that was postponed in the previous section. It is technical and, to help the reader, we decompose in 3 steps, namely Lemma 2.10, Lemma 2.11 and the final argument.

Lemma 2.10. *Under the assumptions of Proposition 2.9, if $t \in \partial X$ then*

$$b \not\preceq t \Leftrightarrow bt = t'b \text{ for some } t' \in S.$$

Proof. Assume $b \not\preceq t$. Set $Y = X \cup \{t\}$. Under the assumptions of Proposition 2.9, we have $b\Delta_Y^{\varepsilon(Y)} \succeq b$. By Proposition 2.5, we deduce that $b\Delta_Y^{\varepsilon(Y)}b^{-1} = \Delta_{Y'}^{\varepsilon(Y)}$ and $bA_Yb^{-1} = A_{Y'}$ for some subset Y' of S . On the other hand, b is a positive X' -ribbon- X for some subset X' of S . It follows that X' is included in $A_{Y'}$ and, therefore, in Y' . Now, the sets X' and Y' have the same cardinality as X and Y , respectively. Then there exists t' in Y' so that $Y' = X' \cup \{t'\}$. We are going to prove that $bt = t'b$. By Lemma 2.7, we can decompose b as $b = b_1b_2$ with b_2 in A_S^+ and b_1 a positive t' -ribbon- t'' for some t'' in S , so that the left orthogonal splitting of $b^{-1}t'b$ is $b_2^{-1}t''b_2$. By the above argument $b^{-1}t'b$ lies in A_Y , so t'' has to lie in Y and b_2 has to lie in A_Y^+ . But b is reduced- Y . Indeed, we assumed that b is reduced- X and that $b \not\preceq t$. This imposes $b_2 = 1$, $b = b_1$ and $bt'' = t'b$ for some t'' in Y . Finally we already have $X'b = bX$. Since t' does not belong to X' , It follows that t'' cannot lie in X . Thus $t'' = t$ and we are done.

Conversely, Assume $bt \succeq b$, then b is a positive ribbon- $\{t\}$. Since it is a positive ribbon- X , it is also a positive ribbon- Y . Denote by $Y(t)$ the irreducible component of Y that contains t . Since t lies in $\partial(X)$, $Y(t)$ contains some element of X . By

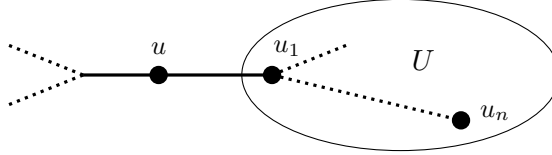
Proposition 2.5 (i)(b) if t is a right-divisor of b then so are all the element of $Y(t)$. But b is reduced- X . Thus t does not right-divide b . \square

Note that we showed the above result without using the assumption: b is a positive ribbon- X^\perp . This hypothesis is then useless for Lemma 2.10.

Lemma 2.11. *Under the assumptions of proposition 2.9, we have*

$$\text{Supp}(b) = S.$$

Proof. By assumption $b \neq 1$, so its support is not empty. Assume by contradiction that $\text{Supp}(b) \neq S$. Let U be an indecomposable component of $\text{Supp}(b)$. Fix u in ∂U and set $V = \text{Supp}(b) \setminus U$. By hypothesis u does not lie in $\text{Supp}(b)$. Then, u does not right-divide b . We claim that bub^{-1} lies in A_S^+ . Indeed, b is a positive ribbon- $X \cup X^\perp$ so if u belongs to $X \cup X^\perp$ there is nothing to say; if u lies in ∂X , then $bu \succeq b$ by Lemma 2.10. Now, the set U is an indecomposable component of $\text{Supp}(b)$, then each element of U commute with each element of V and we can write $b = b_2 b_1$ with $b_1 \in A_U^+$, and $b_2 \in A_V^+$. Write $b_1 = b'_1 s$ with $s \in U$. Since $U \cup \{u\}$ is indecomposable, there exists $u_1, \dots, u_n \in U$ such that $u_0 = u$, $u_n = s$ and $m_{u_i u_{i+1}} > 2$. Up to replacing s by some u_i with $i < n$, we can assume that b has no right-divisor among u_1, \dots, u_{n-1} .



Set $U' = \{u, u_1, \dots, u_{n-1}\}$. Let u_i lies in U' . If u_i does not lies in $X \cup X^\perp$, then u_i belongs to ∂X and, by Lemma 2.10, $bu_i = v_i b$ for some v_i in S . On the other hand b is a positive ribbon- $(X \cup X^\perp)$ therefore b is also a positive ribbon- U' . The graph $\Gamma_{U'}$ is connected by definition, u_n lies in $\partial U'$ and right-divides b . Then by Proposition 2.5, the positive elementary ribbon d_{U', u_n} right-divides b . Applying Lemma 2.6, we get that U' is contained in the support of b . Hence u belongs to $\text{Supp}(b)$, a contradiction. So $\text{Supp}(b) = S$. \square

We are now ready to prove Lemma 2.8

proof of Lemma 2.8. Let $s \in S \setminus X$, and set $Y = S \setminus \{s\}$. Write $b = b_1 b_2$ with $b_2 \in A_Y^+$ and b_1 reduced- Y . By Lemma 2.11, we have $\text{Supp}(b) = S$. Since b_2 lies in A_Y^+ , it follows that $b_1 \neq 1$. In addition, b_1 is reduced- Y . Then, s has to right-divide b_1 . We have $b\Delta_Y^2 b^{-1} = b_1 \Delta_Y^2 b_1^{-1}$. According to the assumptions of Proposition 2.9, we have $b\Delta_Y^2 = zb$ for some z in A_S^+ . Indeed, if $\varepsilon(Y) = 1$ then $b\Delta_Y = z_1 b$ for some $z_1 \in A_S^+$. Therefore $b\Delta_Y^2 = z_1 b\Delta_Y = z_1^2 b$. By Proposition 2.5, we deduce that b_1 is a Y' -ribbon- Y for some $Y' \subseteq S$ and $b = b_1 b_2 = b'_2 b_1$ with $b'_2 \in A_{Y'}^+$. Since s right-divides b_1 , it has also to right-divide b . \square

2.5. When Γ_S is not connected. In Theorem 0.1 we consider irreducible Artin-Tits group A_S of spherical type. Here we extend the theorem to any spherical type Artin-Tits group A_S .

Theorem 2.12. *Let A_S be an Artin-Tits group of spherical type. Denote the indecomposable components of S by S_1, \dots, S_n . Let A_X be a standard parabolic subgroup of A_S and set $X_i = X \cap S_i$ for all i . Set*

$$I = \{1 \leq i \leq n \mid X_i \neq S_i, \Delta_{S_i} \in DZ_{A_{S_i}}(A_{X_i}) \text{ and } \Delta_{S_i} \notin Z(A_{S_i})\}.$$

$$J = \{1 \leq i \leq n \mid X_i \neq S_i, \text{ and } i \notin I\}.$$

Finally, set $S_I = \bigcup_{i \in I} S_i$ and $S_J = \bigcup_{i \in J} S_i$. Then we have

$$DZ_{A_S}(A_X) = A_X \times QZ(A_{S_I}) \times Z(A_{S_J}).$$

Proof. For any direct product of groups $G = G_1 \times \dots \times G_n$ and a subgroup H of G we have

$$Z_G(H) = Z_{G_1}(H_1) \times \dots \times Z_{G_n}(H_n).$$

where $H_i = H \cap G_i$ for each i . Here, $A_S = A_{S_1} \times \dots \times A_{S_n}$ and $A_X \cap A_{S_i} = A_{X_i}$. Now by Theorem 0.1, if i lies in I , then $DZ_{A_{S_i}}(A_{X_i}) = A_{X_i} \times QZ(A_{S_i})$; if i lies in J then $DZ_{A_{S_i}}(A_{X_i}) = A_{X_i} \times Z(A_{S_i})$. In addition, if i is neither in I nor in J , then $X_i = S_i$ and $DZ_{A_{S_i}}(A_{X_i}) = A_{X_i}$. So, we deduce that

$$\begin{aligned} DZ_{A_S}(A_X) &= Z_{A_S}(\prod_{i=1}^n Z_{A_{S_i}}(A_{X_i})) = \prod_{i=1}^n DZ_{A_{S_i}}(A_{X_i}) = \\ &= \prod_{i \in I} (A_{X_i} \times QZ(A_{S_i})) \times \prod_{i \in J} (A_{X_i} \times Z(A_{S_i})) \times \prod_{i \notin I \cup J} A_{X_i} = \\ &= \prod_{i=1}^n A_{X_i} \times \prod_{i \in I} QZ(A_{S_i}) \times \prod_{i \in J} Z(A_{S_i}). \end{aligned}$$

But $\prod_{i=1}^n A_{X_i} = A_X$, $\prod_{i \in I} QZ(A_{S_i}) = QZ(A_{S_I})$ and $\prod_{i \in J} Z(A_{S_i}) = Z(A_{S_J})$. So the equality holds. \square

2.6. Application the subgroup conjugacy problem. Given a group G and a subgroup H of G , the subgroup conjugacy problem for H is solved by finding an algorithm that determines whether any two given elements of G are conjugated by an element of H . In this section, we focus on Artin-Tits groups of type B or D and use Theorem 0.1 and [22, Theorem 1.1] to reduce the subgroup conjugacy problem for their irreducible standard parabolic subgroups to an instance of the simultaneous conjugacy problem. We follow the strategy used in [12] to solve the subgroup conjugacy problem for irreducible standard parabolic subgroups of an Artin-Tits group of type A . The simultaneous conjugacy problem is solved for Artin-Tits groups of type A in [20] (see also [21]), but the result and its proof can be generalized verbatim to all Artin-Tits groups of spherical type, in particular to Artin-Tits groups of type B or type D . Hence, we obtain a solution to the subgroup conjugacy problem for irreducible standard parabolic subgroups of Artin-Tits groups of type B and D .

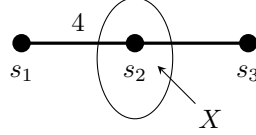
Let us recall [22, Theorem 1.1] and [12, Theorem 2.13].

Theorem 2.13 ([22], Theorem 1.1). *Let A_S be an Artin-Tits group of spherical type such that $\Gamma_S = A_k$ ($k \geq 1$), $\Gamma_S = B_k$ ($k \geq 2$) or $\Gamma_S = D_k$ ($k \geq 4$). Let $X \subseteq S$ such that Γ_X is connected. Then $Z_{A_S}(A_X)$ is generated by*

$$X^\perp \cup \{\Delta_Y \in Z_{A_S}(A_X) \mid X \subseteq Y\} \cup \{\Delta_Y \Delta_{Y'} \in Z_{A_S}(A_X) \mid X \subseteq Y, X \subseteq Y'\}.$$

Note that in the third set, we can restrict the pair (Y, Y) to those so that neither Δ_Y nor $\Delta_{Y'}$ belong to $Z_{A_S}(A_X)$. In the sequel, we denote the obtained generating set by $\Upsilon(X)$.

Example 2.14. Consider $S = \{s_1, s_2, s_3\}$ with A_S of type B_3 as below. Set $X = \{s_2\}$.



We have $X^\perp = \emptyset$ and $Z_{A_S}(A_X)$ is generated by $\Upsilon(X) = \{s_2, \Delta_{\{s_1, s_2\}}, \Delta_S, \Delta_{s_2, s_3}^2\}$.

Theorem 2.15 ([12], Theorem 2.13). *Let G be a group and H be a subgroup such that $DZ_G(H) = Z(G) \cdot H$. Suppose further that $Z_G(H)$ is generated by a set $\{g_1, \dots, g_n\}$. Then for $x, y \in G$, the following are equivalent:*

- (i) *there exists $c \in H$ such that $y = c^{-1}xc$.*
- (ii) *there exists $z \in G$ such that*
 - (a) *$y = z^{-1}xz$, and*
 - (b_i) *$g_i = z^{-1}g_i z$ for all $1 \leq i \leq n$.*

Corollary 2.16. *Let A_S be an Artin-Tits group of type B_k ($k \geq 2$) or D_k ($k \geq 4$). Let $X \subseteq S$ be such that Γ_X is connected. In case Γ_S is of type D_{2k+1} , assume $\{s_2, s_{2'}, s_3\}$ is not included in X with the notations of Figure 5. For any pair (x, y) of elements of A_S , the following are equivalents:*

- (i) *there exists $c \in A_X$ such that $y = c^{-1}xc$.*
- (ii) *there exists $z \in A_S$ such that*
 - (a) *$y = z^{-1}xz$,*
 - (b) *$g = z^{-1}gz$ for all g in $\Upsilon(X)$.*

Proof. By Theorem 0.1 and Proposition 2.1 we have $DZ_G(A_X) = Z(A_S) \times A_X$. So we are in position to apply Theorem 2.15. \square

3. THE NON SPHERICAL TYPE CASES

We turn now to the proof of Theorem 0.3 that is concerned with Artin-Tits groups that are not of spherical type. Our main argument is Proposition 3.1. Indeed, In [17] the second author stated several conjectures, that are proved to hold for Artin-Tits groups of various types. Our proof is based on these conjectures.

Proposition 3.1. (i) *Let A_S be an Artin-Tits group. Assume A_S has the property (\otimes) stated in [17], then for any X included in S one has*

- (a) *If A_X is of spherical type, then for any positive integer k ,*

$$Z_{A_S}(\Delta_X^{2k}) = N_{A_S}(A_X).$$

- (b) *if A_X is of spherical type and there is no Y of spherical type and containing X , then*

$$QZ_{A_S}(A_X) = QZ(A_X) \text{ and } N_{A_S}(A_X) = A_X$$

- (c) *If A_X is irreducible and not of spherical type, then*

$$QZ_{A_S}(A_X) = A_{X^\perp} \text{ and } N_{A_S}(A_X) = A_{X \cup X^\perp}$$

- (ii) [15, 16, 17] *If A_S is of spherical type, of FC type, of large type or of 2-dimensional type. Then, A_S has the Property $(*)$.*

Proof. (i) Conjecture $(*)$ implies that $N_{A_S}(A_X) = A_X \cdot QZ_{A_S}(A_X)$ and that $QZ_{A_S}(A_X)$ is the subgroup of A_S generated by the set of positive X -ribbons- X (see [17]). If A_X is irreducible and not of spherical type, then the set of elementary positive X -ribbons is equal to X^\perp . Moreover all the elements of X^\perp are X -ribbons- X . So $QZ_{A_S}(A_X) = A_{X^\perp}$ and Point (c) holds. Assume A_X is of spherical type. Fix a positive integer k . If g lies in $Z_{A_S}(\Delta_X^{2k})$, then in particular $g^{-1}\Delta_X^{2k}g$ belongs to A_X . Property $(*)$ imposes that that g belongs to $A_X \cdot QZ_{A_S}(A_X)$, that is to $N_{A_S}(A_X)$. Conversely, $A_X \cdot QZ_{A_S}(A_X)$ is included in $Z_{A_S}(\Delta_X^{2k})$ because both A_X and $QZ_{A_S}(A_X)$ have to fix the center of A_X , which contains Δ_X^2 . So Point (a) holds. Finally, if there is no Y of spherical type and containing X , then the elementary positive ribbons $d_{X,t}$ are the elements $\Delta_{X(t)}$ with t in X (see Definition 2.3). It follows that $QZ_{A_S}(A_X)$ is included in A_X and is, therefore, equal to $QZ(A_X)$. Since $N_{A_S}(A_X) = A_X \cdot QZ_{A_S}(A_X)$, we deduce that $N_{A_S}(A_X) = A_X$. Hence Point (b) holds. \square

In the sequel we first extend Conjecture 0.2 to the context of non irreducible parabolic subgroup (see Conjecture 3.2). Then we prove that Conjecture 3.2 holds for any Artin-Tits which possesses the property $(*)$ (see Theorem 3.4). Considering Proposition 3.1 (ii), this will prove Theorem 0.3.

Conjecture 3.2. Let A_S be an irreducible Artin-Tits group and X be included in S . Let X_s be the union of the irreducible components of X that are of spherical type, and X_{as} be the union of the other irreducible components of X . Then,

$$DZ_{A_S}(A_X) = Z_{A_S}(Z_{A_{X_{as}^\perp}}(A_{X_s}))$$

- (i) Assume X_s is empty. Then

$$DZ_{A_S}(A_X) = Z_{A_S}(A_{X^\perp}).$$

- (ii) Assume A_X is of spherical type. Let A_T be the smallest standard parabolic subgroup of A_S that contains $Z_{A_S}(A_X)$.
 (a) If T is of spherical type then

$$DZ_{A_S}(A_X) = DZ_{A_T}(A_X).$$

- (b) If T is not of spherical type then

$$DZ_{A_S}(A_X) = A_X.$$

Proposition 3.3. *Let A_S be an irreducible Artin-Tits group and X be included in S . Assume A_S has the property $(*)$ stated in [17]. Conjecture 3.2 implies Conjecture 0.2.*

Proof. Consider the notations of Conjecture 0.2. Assume X is irreducible. If X is not of spherical type, then $X = X_{as}$ and X_s is empty. By Proposition 3.1, $Z_{A_S}(A_X) \subseteq QZ_{A_S}(A_X) = A_{X^\perp} \subseteq Z_{A_S}(A_X)$. Therefore $A_{X^\perp} = Z_{A_S}(A_X)$ and $T = X^\perp$. Thus, Conjecture 3.2(i) implies Conjecture 0.2(i). In the case X is of spherical type, there is nothing to prove. \square

Theorem 3.4. *Let A_S be an irreducible Artin-Tits group. If A_S has the property $(*)$ stated in [17], then Conjecture 3.2 holds.*

In order to prove Theorem 3.4 We need some preliminary results. In the sequel, we assume A_S is an irreducible Artin-Tits group that has the property (\otimes) stated in [17]. We fix a standard parabolic subgroup A_X with $X \subseteq S$. By X_s we denote the union of the irreducible components of X that are of spherical type. By X_{as} we denote the union of the other irreducible components of X . By definition X_s is included in X_{as}^\perp . We set

$$\Upsilon = \{Y \subseteq S \mid X_s \subseteq Y; \text{ and } A_Y \text{ is of spherical type.}\}$$

Let A_T be the smallest standard parabolic subgroup of A_S that contains $Z_{A_S}(A_{X_s})$.

Lemma 3.5. $Z_{A_S}(A_X) = Z_{A_{X_{as}^\perp}}(A_{X_s})$.

Proof. We have $A_X = A_{X_s} \times A_{X_{as}}$. Therefore $Z_{A_S}(A_X) = Z_{A_S}(A_{X_s}) \cap Z_{A_S}(A_{X_{as}})$. Let X_1, \dots, X_k be the irreducible components of X_s . Then $X_{as}^\perp = X_1^\perp \cap \dots \cap X_k^\perp$. On the other hand, $A_{X_{as}} = A_{X_1} \times \dots \times A_{X_k}$ and $Z_{A_S}(A_{X_{as}}) = Z_{A_S}(A_{X_1}) \cap \dots \cap Z_{A_S}(A_{X_k})$. By Proposition 3.1, $Z_{A_S}(A_{X_i}) = QZ_{A_S}(A_{X_i}) = A_{X_i^\perp}$ for each component X_i . Therefore $Z_{A_S}(A_{X_{as}}) = A_{X_1^\perp} \cap \dots \cap A_{X_k^\perp} = A_{X_1^\perp \cap \dots \cap X_k^\perp} = A_{X_{as}^\perp}$. But, A_{X_s} is included in $A_{X_{as}^\perp}$. Thus, $Z_{A_S}(A_{X_s}) \cap Z_{A_S}(A_{X_{as}}) = Z_{A_{X_{as}^\perp}}(A_{X_s})$. \square

Lemma 3.6. *The set Υ is not empty and all its elements are contained in T . Moreover, T belongs to Υ if and only if T is of spherical type. In this case, T is the unique maximal element of Υ .*

Proof. X_s is contained in Υ , so the latter is not empty. Moreover, X_s is included in T . Therefore the latter belong to Υ if and only if it is of spherical type. Finally if Y belongs to Υ , then Δ_Y^2 belongs to $Z_{A_S}(A_Y)$, and therefore to $Z_{A_S}(A_X)$. Thus, Y is included in T . Hence, if T belongs to Υ , it is its unique maximal element. \square

Lemma 3.7. *Assume Y is maximal in Υ for the inclusion. Then,*

$$DZ_{A_S}(A_{X_s}) \subseteq DZ_{A_Y}(A_{X_s})$$

Proof. Assume g belongs to $DZ_{A_S}(A_{X_s})$. The element Δ_Y^2 lies in $Z(A_Y)$. Since X_s is included in Y , it follows that Δ_Y^2 lies in $Z_{A_S}(A_{X_s})$, and $g\Delta_Y^2g^{-1} = \Delta_Y^2$. By Proposition 3.1(i)(a), g belongs to the subgroup $N_{A_S}(A_Y)$. But Y is maximal in Υ . By Proposition 3.1(i)(b), $N_{A_S}(A_Y) = A_Y$. Thus $DZ_{A_S}(A_{X_s}) = Z_{A_S}(Z_{A_S}(A_{X_s})) \cap A_Y = Z_{A_Y}(Z_{A_S}(A_{X_s})) \subseteq Z_{A_Y}(Z_{A_Y}(A_{X_s})) = DZ_{A_Y}(A_{X_s})$. \square

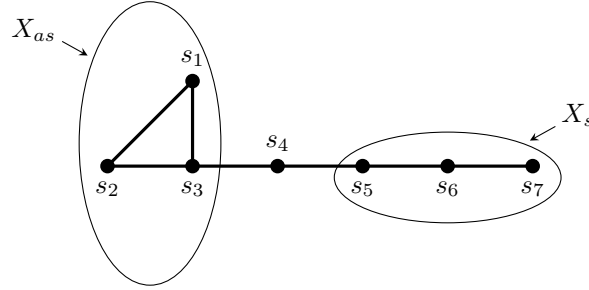
We can now prove Theorem 3.4

Proof of Theorem 3.4. By Lemma 3.5, we have $Z_{A_S}(A_X) = Z_{A_{X_{as}^\perp}}(A_{X_s})$. It follows that $DZ_{A_S}(A_X) = Z_{A_S}(Z_{A_{X_{as}^\perp}}(A_{X_s}))$. When X_s is empty, we have $X_{as} = X$, $A_{X_s} = \{1\}$. So $Z_{A_{X_{as}^\perp}}(A_{X_s}) = Z_{A_{X^\perp}}(\{1\}) = A_{X^\perp}$. Therefore, $DZ_{A_S}(A_X) = Z_{A_S}(A_{X^\perp})$. Assume for the remaining of the proof that X is of spherical type. Assume, First, that T is of spherical type. By Lemma 3.6, T is maximal in Υ and, by Lemma 3.7, $DZ_{A_S}(A_X) \subseteq DZ_{A_T}(A_X)$. On the other hand, $Z_{A_T}(A_X) = Z_{A_S}(A_X) \cap A_T = Z_{A_S}(A_X)$. We deduce that $DZ_{A_T}(A_X) = Z_{A_T}(Z_{A_S}(A_X)) \subseteq DZ_{A_S}(A_X)$. Hence, $DZ_{A_S}(A_X) = DZ_{A_T}(A_X)$. Assume, finally, that T do not lie in Υ . Let Y be maximal in Υ . By Lemma 3.7, we get $DZ_{A_S}(A_X) \subseteq DZ_{A_Y}(A_X)$. If $Y = X$, then $A_X \subseteq DZ_{A_S}(A_X) \subseteq DZ_{A_X}(A_X) = A_X$ and we are done. So, assume $X \subsetneq Y$. The group A_Y is of spherical type. Applying Theorem 0.1, we get that $DZ_{A_Y}(A_X) \subseteq QZ(A_Y) \times A_X$. Since A_X is included in $DZ_{A_S}(A_X)$, the group A_X

is equal to $DZ_{A_S}(A_X)$ if and only if $DZ_{A_S}(A_X) \cap QZ(A_Y) = \{1\}$. Assume this is not the case. Then, there exists $k > 0$ so that Δ_Y^k lies in $DZ_{A_S}(A_X)$. We can assume without restriction that k is even. Since Y lies in Υ and T does not, they are distinct. It follows from the definition of T that there exists g in $Z_{A_S}(A_X)$ which is not in A_Y . But Δ_Y^k lies in $DZ_{A_S}(A_X)$. So we have $\Delta_Y^k g (\Delta_Y^k)^{-1} = g$, and equivalently $g \Delta_Y^k g^{-1} = \Delta_Y^k$. The latter equality imposes that g belongs to $N_{A_S}(A_Y)$ by Proposition 3.1(i)(a). But $N_{A_S}(A_Y) = A_Y$ by Proposition 3.1(i)(b), a contradiction. Hence, $DZ_{A_S}(A_X) = A_X$. \square

Corollary 3.8. *Let A_S be an irreducible Artin-Tits group of FC type, or of large type, or of 2-dimensional type. Then Conjecture 3.2 holds.*

Remark 3.9. *In an (irreducible) Artin-Tits group that is of large type, all standard parabolic subgroups are irreducible. So, Corollary 3.8 provides a complete description of the double centralizer of any standard parabolic subgroup. However, for the other not spherical types in the case both X_s and X_{as} are not empty, the answer is not completely satisfactory. Indeed the double centralizer is not as simple as in the cases where either X_s or X_{as} is empty. For instance, in the following example, we have $Z_{A_S}(A_X) = Z(A_{X_s})$ and $DZ_{A_S}(A_X) = N_{A_S}(A_{X_s}) = QZ_{A_S}(A_{X_s}) \cdot A_{X_s}$*



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